Paul Keyser

Grasping the Limits: As easy as pi

Mathematicians working in the Greek tradition repeatedly addressed certain persistent problems, such as squaring the circle, and successive results often led them to reinterpret the problems. Because the results often amounted to incremental changes in how the problems were perceived, some problems were not so much regarded as solved or no longer relevant, but instead were redefined. How did the under&endash;standing of persistent problems evolve in the light of new results?

Ancient mathematicians generally agreed that infinities were not tractable, and probably even non-existent. Nevertheless, certain problems seemed to lead naturally to various kinds of infinity, which led mathematicians and philosophers to attempt to deal with those problems without introducing infinities. Several puzzles resulted, for example: was there a largest prime number? Did one need to prove separately for each kind of rectangle the incommen&endash;surability of its side and diagonal? What would it mean to measure the area of a circle by means of polygons?

In this paper, I will discuss how the understanding of infinite series gradually evolved, from Zen to Archimedes, in the twin contexts of finding areas and volumes by the "method of exhaustion" and of summing an infinite series of numbers.

Zen had been puzzled to understand how an endless series of finite quantities could reach any finite total or limit (DK A25-26: Aristotle, *Phys*. 6.2 [233a21-32] + 6.9 [239b9-30]; cp. 8.8 [263a4-b8]).

Demokritos (DK B155), Antiphōn (DK B13), and probably Eudoxos, sought to compute the area or volume of curvilinear figures using an endless series of rectilinear figures, e.g., to "square" the circle. This procedure Aristotle rejected as ill-founded (*Phys*. 1.2 [185a14-18], *Soph. El*. 11 [172a3-9]), but Euclid explored and improved (*Elem*. 10.1, 12.2). Archimedes, building upon the work of Demokritos and Eudoxos (*Sphere and Cylinder praef*., *Method praef*.), dealt with limits in *Measurement of the Circle* and *Sphere and Cylinder* 1.1-1.6, and achieved in his *Method* results comparable to those of the integral calculus.

Aristotle, when treating the problem of infinite series (*Phys*. 3.6 [206b3-34]), distin&endash;guished between infinite series of additions and of divisions, finding the latter more tractable. Euclid treated the same pair of problems (*Elem*. 9.35), for an arbitrary number of terms of the series, but did not treat the infinite case. Archimedes (*Quad. Parabol*. 22-23) does explicitly discuss the general case, including the infinity, and derives a general formula for the sum.

David Bostock, *Plato's Theaetetus* (Oxford 1988) 211-222.
Alan C. Bowen, "The exact sciences in Hellenistic times," *Routledge History of Philo&endash;sophy*, v. 2 (1999) 287-319. William Charlton, "Aristotle's potential infinities," *Aristotle's Physics: A collection of essays*, ed. Lindsay Judson (Oxford 1991)

129-150.
A. García de la Sienra, "El Metodo de Arquimedes," *Dianoia* 29 (1983) 53-80.
Andreas Graeser, ed., *Mathematics and Metaphysics in Aristotle* (Bern/ Stuttgart 1987).
Theokritos Kouremenos, *Aristotle on Mathematical Infinity* (Stuttgart 1995).
Richard D. McKirahan, "Zeno," *Cambridge Companion to Early Greek Philosophy*, ed. A. A. Long (Cambridge 1999) 134-158.
Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (MIT 1991) 101-106, 234-236.
G. J. Pendrick, *Antiphon the Sophist* (Cambridge 2001) 108-125, 261-275.
R. Seide, "Zum Problem des geometrischen Atomismus bei Demokrit," *Hermes* 109 (1981) 265-280.
E. S. Stamatis, "Genikeusis enos theorematos tou Archimedous," *Platon* 15 (1963) 165-168.
M. J. White, *The continuous and the discrete* (Oxford 1992) 8-14, 208-209, 293-306.